Darboux Transformation and Exact Solutions in the model of Cylindrically Symmetrical Chiral Field

E.Sh.Gutshabash

St.-Petersburg State University, Russia
e-mail: gutshab@EG2097.spb.edu

The application of the Darboux Transformation method to the integrable model of Cylindrically Symmetrical Chiral field has been considered. The associated linear system of matrix equations has been proposed and the properties of symmetrie for its solutions has been obtained. The necessary form of Darboux Transformation has been found and formal one- and N-soliton solutions have been constructed. With the use of Pohlmayer's Transformation the equation sin-Gordon type have been given and the hypothesis about its integrability has been deduced.

1. It is well known, that all methods of solution of nonlinear integrable equations of modern theoretical and mathematical physics can be divided into two large groups: the direct, when the Lax representation of initial equation is not required, and indirect one, when we have the evident representation of our nonlinear equation as the overdetermined system two (matrix) linear equations on any auxiliary function.

To the first group one can assign the ansatz-method, the Hirota's method and etc. The second group includes such methods as the Backlund transformation, the different versions of the dressing procedures, the Inverse Scattering Transform. Among this group the method of Darboux transformation (DT) has a significant place. The numerous of examples of its advantegeoures use are given in [1]. However the further progress of the theory of nonlinear integrable equations requires that the new and more complex versions of DT are involved .

The aim of this paper is to demonstrate one of this versions by the example of equation cylindrically symmetrical chiral field (σ- model), considered originally in [2]. In this case the method of DT using appear to be rather nontrivial, it calls for a preliminary detailed study of the algebraic properties of the Lax representation.

2. Equation of the model of the cylindrically symmetrical chiral field has the form:

$$\partial_\eta(\alpha g_\xi g^{-1}) + \partial_\xi(\alpha g_\eta g^{-1}) = 0, \tag{1}$$

where $\xi = (t-r)/2$, $\eta = (t+r)/2$ are the cone variables, g is the element of some group Lee $\mathcal{G}$ (or symmetrical spaces), $\alpha = r$.

This equation can be rewritten in two equivalent forms:

$$(g_t g^{-1})_t = (g_r g^{-1})_r + \frac{1}{r} g_r g^{-1}, \tag{$1'$}$$

$$(r g_t g^{-1})_t = (r g_r g^{-1})_r. \tag{$1''$}$$

It should be noted, that appearance of equation (1) is coincides with one of the following equations: one of the versions of the Ernst's equations in the gravitation' theory

[3] (for $\mathcal{G} = SL(2)/SO(2)$), the model of selfdual Yang-Mills's field (model of Zakharov-Manakov) [4] (for $\mathcal{G} = SU[N]$), the model of stationary two-dimensional ferromagnet of Heisenberg possessing variable nominal magnetization [5, 6]; the appearance is encountered in the series of another problems. The significance of the equation (1) is that it is the preimidge more realistic four-dimensional gauge theories.

In [2] this equation was studied to perfect the procedure dressing solutions. Also in paper [7] it has been examined employing the method of the harmonical maps.

It should be notice, that the model (1) becomes (2+1)-dimensional when it is supposed that $r = \sqrt{x^2 + y^2}$, where x, y are the Cartesian coordinates on the plane.

Here we will restrict ourselves to the case of $g \in GL(2, \mathbb{C})$, with g can be parameterized in the form: $g = \sum_{i=1}^{3} g_i \sigma_i$, $\mathbf{g} = (g_1, g_2, g_3)$ is the unit vector with the real components, σ_i are the standard Pauli matrices. Then g posses the properties:

$$g^2 = I, \qquad \det g = -1, \qquad g = g^+, \qquad Tr\, g = 0, \tag{2}$$

where I is unit 2×2 matrices, and symbol g^+ means the hermitian conjugate.

At first we will dwell on the linear version of equation (1). In this case it reduces to:

$$g_{tt} = g_{rr} + \frac{1}{r} g_r, \tag{3}$$

and can be easy solved. We have

$$g = \begin{pmatrix} g_3 & g_- \\ g_+ & -g_3 \end{pmatrix}, \tag{4}$$

where

$$g_i(r,t) = \int dk (C_i^1(k) e^{kt} + C_i^2(k) e^{-kt})(a_i(k) J_0(kr) + b_i(k) N_0(kr)), \tag{5}$$

and $g_+ = g_1 + i g_2$, $g_- = \bar{g}_+$. Here J_0, N_0 are the function of Bessel and Neeman, functions C_i^1, C_i^2, a_i, b_i should be determined from the initial and boundary conditions, and also the condition $\det g = -1$.

Returning to the nonlinear equation (1), one can attract the ansatz of the following specials kind:

$$g = \begin{pmatrix} \cos\chi & e^{-i\Phi} \sin\chi \\ e^{i\Phi} \sin\chi & -\cos\chi \end{pmatrix}, \tag{6}$$

where $\Phi = \Phi(r,t)$, $\chi = \chi(r,t)$ are real functions, which need to determination. Substituting (6) into (1), after simple calculations we obtain the system of nonlinear differential equations:

$$\begin{gathered} (\Phi_{tt} - \Phi_{rr} - \frac{1}{r}\Phi_r) \sin\chi + 2(\Phi_t \chi_t - \Phi_r \chi_r) \cos\chi = 0, \\ 2(\chi_{tt} - \chi_{rr} - \frac{1}{r}\chi_r) = (\Phi_t^2 - \Phi_r^2) \sin 2\chi. \end{gathered} \tag{7}$$

There is interest some special cases of this system. Let $\chi = 0$, then, evidently, $g = \sigma_3$. At $\chi = \pi/2$ on the function Φ we obtain the scalar equation, which coincides on his kind with (3). The same equation arises at $\Phi = 0$ on function χ. Now setting $\Phi = kt$, where k is the free parameter, from the first equation (7) it follows that $\chi = \chi(r)$, and second equation gives:

$$\chi_{rr} + \frac{1}{r}\chi_r = -\frac{k^2}{2}\sin 2\chi. \tag{8}$$

This equation at the properly imagine k after the automodel substitution reduces to the equation Painleve-III.

3. To apply DT method to (1) we write the associated linear system of equation as

$$\Psi_\xi = U\Psi\Lambda_- \quad , \quad \Psi_\eta = V\Psi\Lambda_+, \tag{9}$$

where $\Psi(\xi,\eta,\lambda) \in Mat(2,\mathbb{C})$, $U = g_\xi g^{-1}$, $V = g_\eta g^{-1}$ are left currents (elements algebras Lee g), and diagonal matrices Λ_-, Λ_+, are determined by the expressions:

$$\Lambda_- = \begin{pmatrix} -\frac{r}{\gamma-r} & 0 \\ 0 & \frac{\overline{\gamma}}{\overline{\gamma}-r} \end{pmatrix}, \qquad \Lambda_+ = \begin{pmatrix} \frac{r}{\gamma+r} & 0 \\ 0 & \frac{\overline{\gamma}}{\overline{\gamma}+r} \end{pmatrix}. \tag{10}$$

The function $\gamma(\xi,\eta,\lambda) \equiv \gamma(r,t,\lambda)$ which was introduced in (3), contains so-called "hidden" complex parameter:

$$\gamma = \xi + \eta - \lambda + \sqrt{(\lambda-2\xi)(\lambda-2\eta)} = t - \lambda + \sqrt{(t-\lambda)^2 - r^2}, \tag{11}$$

where $\lambda \in \mathbb{C}$ is the parameter that does not depend on the coordinate and time.

The condition of the compatibility of the system (2) : $\Psi_{\xi\eta} = \Psi_{\eta\xi}$ is equivalent to both of following two relations:

$$U_\eta - V_\xi + [U,V] = 0,$$

which asserts identically and (1). So, equation (1) is quite integrable.

We will study some properties of system (2), including the symmetries.

Matrices Λ_+, Λ_- satisfy the series of useful identities. In order to demonstrate this fact, we introduce also the matrices σ_0 and M:

$$\sigma_0 = \begin{pmatrix} \frac{\gamma}{(\gamma+r)(\gamma-r)} & 0 \\ 0 & -\frac{\overline{\gamma}}{(\overline{\gamma}-r)(\overline{\gamma}+r)}) \end{pmatrix}, \tag{12}$$

$$M = \sigma_0\Lambda_+^{-1}\Lambda_-^{-1} = \begin{pmatrix} \frac{r^2}{\gamma} & 0 \\ 0 & \overline{\gamma} \end{pmatrix}. \tag{13}$$

By the direct calculation one can check, that the following identities are true:

$$\Lambda_+ = \Lambda_+\Lambda_- - r\sigma_0, \qquad \Lambda_{-\eta} = \sigma_0, \tag{14}$$

$$\Lambda_- = \Lambda_+\Lambda_- + r\sigma_0, \qquad \Lambda_{+\xi} = \sigma_0, \tag{15}$$

$$M\Lambda_+ = M - r\Lambda_+, \qquad M_\eta = 2\Lambda_+, \tag{16}$$

$$M\Lambda_- = M + r\Lambda_-, \qquad M_\xi = 2\Lambda_-, \tag{17}$$

$$\Lambda_+ = I + rM^{-1}, \qquad \Lambda_- = I + rM^{-1}, \tag{18}$$

$$\Lambda_{-\xi} = -\frac{2}{r}\Lambda_-^3 + \frac{3}{r}\Lambda_-^2 - \frac{1}{r}\Lambda_-, \tag{19}$$

$$\Lambda_{+\eta} = \frac{2}{r}\Lambda_+^3 - \frac{3}{r}\Lambda_+^2 + \frac{1}{r}\Lambda_+. \tag{20}$$

Series of relations (14)-(20) it is naturally to call the kinematical connections. Using them it is not difficult to obtain some more useful two connection, written in terms matrices M:

$$\begin{gathered} M_{\xi\xi} = -\frac{1}{2r}M_\xi^3 + \frac{3}{2r}M_\xi^2 - \frac{1}{r}M_\xi, \\ \\ M_{\eta\eta} = \frac{1}{2r}M_\eta^3 - \frac{3}{2r}M_\eta^2 + \frac{1}{r}M_\eta. \end{gathered} \tag{21}$$

Moreover, for matrices M we have:

$$M_{\xi\eta} = -2(I - rM^{-1})^{-1}(I + rM^{-1})M^{-1}, \tag{22}$$

that is it satisfy to nonlinear wave equation and

$$[M, M_\xi] = [M, M_\eta] = [M_\xi, M_\eta] = [M_{\xi\xi}, M_{\eta\eta}] = \cdots = 0. \tag{23}$$

Now we will study the properties of symmetry of the solution of the equation (1). From the determination matrices g it follows, that the current must meet the condition:

$$U = \sigma_2 \overline{U} \sigma_2. \tag{24}$$

Let $\hat{\tau}$ - the operation of the transposition of the sheets of Riemann surface Γ for function $\gamma : \gamma(\hat{\tau}(\lambda)) = r^2/\gamma(\lambda)$. Then

$$\Lambda_\pm(\hat{\tau}(\lambda)) = \sigma_2 \overline{\Lambda}_\pm(\lambda)\sigma_2. \tag{25}$$

From the expressions (9), (24)-(25) it follows

$$\Psi(\lambda) = \sigma_2 \overline{\Psi}(\hat{\tau}(\lambda))\sigma_2. \tag{26}$$

Since $g = g^{-1}$, and taking into account (9), we obtain:

$$\sigma_2 g \Psi(\lambda)\sigma_2 = \overline{\Psi}(\lambda)\sigma_3. \tag{27}$$

Relations (25)-(26), which are the consequence of the properties of function $\gamma(\xi, \eta)$, are the whole set of the involutions of the problem (9), so that all solutions determined below must be satisfy it.

4. Let's consider the construction the exact solutions of the model (1). In view of, both equation of the Lax pair should be covariant relatively DT and taking into account

the second equality (16)-(17), we rewriting the associated linear system (9) in the more symmetrical form:

$$\Psi_\xi = \frac{1}{2} U \Psi M_\xi, \qquad \Psi_\eta = \frac{1}{2} V \Psi M_\eta. \tag{28}$$

Check the covariance (28) relatively the matrix DT:

$$\tilde{\Psi} = \Psi - L_1 \Psi M, \tag{29}$$

where $L_1 = \Psi_1 M_1^{-1} \Psi_1^{-1}$, Ψ_1 is some fixed solution of equation (9). So, we require that (9) holds its form:

$$\tilde{\Psi}_\xi = \frac{1}{2} \tilde{U} \tilde{\Psi} M_\xi, \quad \tilde{\Psi}_\eta = \frac{1}{2} \tilde{V} \tilde{\Psi} M_\eta, \tag{30}$$

where $\tilde{U} = \tilde{g}_\xi \tilde{g}^{-1}$, $\tilde{V} = \tilde{g}_\eta \tilde{g}^{-1}$ are the "dressing" currents.

Substituting the ansatz (29) in the first equation of the system (30), we find:

$$\tilde{U} = U + r L_{1\xi} - 2L_1, \tag{31}$$

$$\tilde{U} = L_1 U L_1^{-1} + L_{1\xi} L_1^{-1}, \tag{32}$$

where U is the initial solution of equation (1). Note also, that the check of the equivalence this relations gives identity (19).

One can write formulas (31) in more conventional form:

$$\tilde{U} = U + r \Psi_1 [\Psi_1^{-1} \Psi_{1\xi}, M_1^{-1}] \Psi_1^{-1} - \Psi_1 M_1^{-1} M_{1\xi} \Psi_1^{-1}. \tag{33}$$

Analogously, from the second equation (28) we will have:

$$\tilde{V} = V - r L_{1\eta} - 2L_1, \tag{34}$$

$$\tilde{V} = L_1 V L_1^{-1} + L_{1\eta} L_1^{-1}, \tag{35}$$

where V is the initial solution equation (1) and the check of the equivalence (34) and (35) gives identities (20). Besides,

$$\tilde{V} = V - r \Psi_1 [\Psi_1^{-1} \Psi_{1\eta}, M_1^{-1}] \Psi_1^{-1} - \Psi_1 M_1^{-1} M_{1\eta} \Psi_1^{-1}. \tag{36}$$

But the solutions (33), (36), which are the dressing relations not yet desired solutions (1). Indeed, from (33) and (36) we have:

$$Tr\tilde{U} \neq 0 \ \ , \ \ Tr\tilde{V} \neq 0. \tag{37}$$

This inequalities contradict the condition (2). In order to remedy the situation it should be noted that in equation (1) the currents U, V are not single-valued but are determined with an accuracy of substitution:

$$U \to U - 1/2[\gamma_\xi - r(\ln \overline{\gamma})_\xi] I, \qquad V \to V + 1/2[\gamma_\eta + r(\ln \overline{\gamma})_\eta] I. \tag{38}$$

Therefore, the final expressions for the currents can be written as:

$$\tilde{U} = U + r\Psi_1[\Psi_1^{-1}\Psi_{1\xi}, M_1^{-1}]\Psi_1^{-1} - \Psi_1 M_1^{-1} M_{1\xi}\Psi_1^{-1} - \frac{1}{2}[-2(\ln r)_\xi + (\ln\frac{\gamma}{\bar{\gamma}})_\xi]I, \qquad (39)$$

$$\tilde{V} = V - r\Psi_1[\Psi_1^{-1}\Psi_{1\eta}, M_1^{-1}]\Psi_1^{-1} - \Psi_1 M_1^{-1} M_{1\eta}\Psi_1^{-1} - \frac{1}{2}[-2(\ln r)_\eta + (\ln\frac{\gamma}{\bar{\gamma}})_\eta]I. \qquad (40)$$

Using, for examples (39), one can obtain the formal expression for the initial matrices $\tilde{g}$:

$$\tilde{g} \equiv g[1] = T\exp\left(\int^\xi A(\xi,\eta)d\xi\right)g^{(1)}, \qquad (41)$$

where $g^{(1)}$ is initial solution of equation (1), $A(\xi,\eta)$ is the right-side of relation (39) and symbol T means T - ordered exponent. Formulas (41) gives the one soliton solution of the equation (1).

Now we will construct N-soliton solution of the problem. For this it should be notice, that

$$\begin{gathered} \Psi[1] = \Psi - L_1\Psi M, \\ \Psi[2] = \Psi[1] - L_2\Psi[1]M = \Psi - (L_1+L_2)\Psi M + L_2L_1\Psi M^2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Psi[N] = \Psi + T_1\Psi M + \ldots\ldots\ldots\ldots + T_N\Psi M^N, \end{gathered} \qquad (42)$$

where $L_i = \Psi_i M_i \Psi_i^{-1},\ M_i = M(\lambda_i),\ \lambda_1, \ldots.\lambda_N$ is the set of the fixed complex parameters. Then the coefficients $T_i,\ i = 1, N$, can be determined from the conditions:

$$\Psi[N]_{|\Psi=\Psi_i, M=M_i} = 0, \qquad (43)$$

which gives us the system of linear equations:

$$\begin{gathered} T_1\Psi_1M_1 + T_2\Psi_1M_1^2 + \ldots.. + T_N\Psi_1M_1^N = -\Psi_1, \\ T_1\Psi_2M_2 + T_2\Psi_2M_2^2 + \ldots.. + T_N\Psi_2M_2^N = -\Psi_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ T_1\Psi_NM_N + T_2\Psi_NM_N^2 + \ldots.. + T_N\Psi_NM_N^N = -\Psi_N. \end{gathered} \qquad (44)$$

On the another side, accordingly (39)

$$\begin{gathered} U[1] = U + rL_{1\xi} - 2L_1 + Q_1, \\ U[2] = U[1] + rL_{2\xi} - 2L_2 + Q_2 = U + r(L_{1\xi} + L_{2\xi}) - 2(L_1+L_2) + Q_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ U[N] = U + r\sum_{i=1}^N L_{i\xi} - 2\sum_{i=1}^N L_i + \sum_{i=1}^N Q_i, \end{gathered} \qquad (45)$$

where $Q_i = \left[\ln(\gamma(\lambda_i)/r^2\gamma(\overline{\lambda}_i))\right]_\xi$. Thus, for the Nth dressed current we will have:

$$U[N] = U + rT_{1\xi} - 2T_1 + \sum_{i=1}^{N} Q_i. \tag{46}$$

From this equation we obtain N-soliton solution:

$$g[N] = T\exp\left(\int^{\xi} [U + rT_{1\xi} - 2T_1 + \sum_{i}^{N} Q_i] d\xi\right) g^{(N)}. \tag{47}$$

The approach which is equivalent developed above, can be realized by the transition in (9) to the Hermitian objects. Putting $\Phi = \Psi^*$, we can rewritten (9) as

$$\Phi_\xi = -\overline{\Lambda}_- \Phi U \quad , \quad \Phi_\eta = -\overline{\Lambda}_+ \Phi V. \tag{48}$$

The condition of the compatibility of this system again gives equation (1) and, consequently, we also can obtain corresponding dressing relations, which are equivalent to ones already find. In this case for DT it should be taken ($\tilde{\Phi} \equiv \Phi[-1]$):

$$\tilde{\Phi} = \Phi - M_1^* \Phi L_1^*. \tag{49}$$

5. Taking into account (2), the equation (1) reduces to

$$\mathbf{g}_{tt} - \frac{1}{r}(r\mathbf{g})_r + (\mathbf{g}_t^2 - \mathbf{g}_r^2)\mathbf{g} = 0. \tag{50}$$

It can be obtain by minimization an action with the Lagrangian density

$$\mathcal{L} = (1/4) r Tr(g_\xi g_\eta) = (1/4) r Tr(g_t^2 - g_r^2) = (1/2) r(\mathbf{g}_t^2 - \mathbf{g}_r^2).$$

This allow us to introduce the Hamiltonian description of our system with Hamiltonian (μ - Lagrange multiplicator):

$$H = \frac{1}{2}\int drr \left[(\mathbf{p}^2 + \mathbf{g}_r{}^2) - \mu(\mathbf{g}^2 - 1)\right], \tag{51}$$

where $\mathbf{p} = \mathbf{g}_t$, moreover, the phase space of the model forms by variables $\mathbf{g}(r,t)$, $\mathbf{p}(r,t)$.

Appropriate Hamiltonian equations have the form:

$$\begin{aligned} \mathbf{g}_t &= \frac{1}{r}\frac{\delta H}{\delta \mathbf{p}} = \{H, \mathbf{g}\}, \\ \mathbf{p}_t &= -\frac{1}{r}\frac{\delta H}{\delta \mathbf{g}} = \{H, \mathbf{p}\}. \end{aligned} \tag{52}$$

Here the Poissonian structure on the phase space, as in [8], is introduced with the help of the fundamental brackets, obtained with the account Dirac's connections $g_i g_i - 1 = 0$, $g_i g_{ri} = g_i g_{ti} = 0$:

$$\{g_i(r), g_k(r')\} = 0,$$

$$\{p_i(r), p_k(r')\} = -\left[p_i(r)g_k(r) - p_k(r)g_i(r)\right]\delta(r-r'),$$
$$\{p_i(r), g_k(r')\} = \left[\delta_{ik} - g_i(r)g_k(r)\right]\delta(r-r'). \tag{53}$$

Then for two arbitrary smooth functionals F and G we will have:

$$\{F, G\} = \int dr \Big\{ \left[\frac{\delta F}{\delta \mathbf{p}}\frac{\delta G}{\delta \mathbf{g}} - \frac{\delta F}{\delta \mathbf{g}}\frac{\delta G}{\delta \mathbf{p}}\right] +$$
$$+\left(\frac{\delta G}{\delta \mathbf{p}}\mathbf{g}\right)\left[\left(\frac{\delta F}{\delta \mathbf{g}}\mathbf{g}\right) - \left(\frac{\delta F}{\delta \mathbf{p}}\mathbf{p}\right)\right] + \left(\frac{\delta F}{\delta \mathbf{p}}\mathbf{g}\right)\left[\left(\frac{\delta G}{\delta \mathbf{p}}\mathbf{p}\right) - \left(\frac{\delta G}{\delta \mathbf{g}}\mathbf{g}\right)\right]\Big\}. \tag{54}$$

In terms the currents Hamiltonian takes the form:

$$H = \int drrTr\left\{U^2 + V^2\right\}. \tag{55}$$

It can checked by direct calculation, that the equation (1) is equivalent to two ones:

$$U_\eta = -\frac{1}{2}\left[U, V\right] - \frac{1}{2r}(U - V),$$
$$V_\xi = \frac{1}{2}\left[U, V\right] - \frac{1}{2r}(U - V). \tag{56}$$

Besides that, in terms of the variables U, V we have one more useful entry (1) having evident kind of the conservation law:

$$\left(U + \int^\eta (\ln r)_{\eta'} U d\eta'\right)_\eta + \left(V + \int^\xi (\ln r)_{\xi'} V d\xi'\right)_\xi = 0. \tag{57}$$

6. Of some interest is the question about the correspondence of model (1) and equation of the type sin-Gordon. To obtain appropriate equation let us invoke the transformation of Pohlmayer [9]. Set

$$\mathbf{g}_{\xi\xi} = c_1\mathbf{g} + c_2\mathbf{g}_\xi + c_3\mathbf{g}_\eta \quad , \quad \mathbf{g}_{\eta\eta} = c_4\mathbf{g} + c_5\mathbf{g}_\xi + c_6\mathbf{g}_\eta, \tag{58}$$

where $c_i = c_i(\xi, \eta)$, $i = 1, 6$ are real functions to be determined, and introduce also the function $f = f(u)$:

$$f(u(\xi, \eta)) = \mathbf{g}_\xi\mathbf{g}_\eta. \tag{59}$$

Note, that, generally speaking, without loss the community, we can suppose, that

$$\mathbf{g}_\xi^2 = \mathbf{g}_\eta^2 = 1. \tag{60}$$

Using (56)-(57), we find:

$$\mathbf{g}_{\xi\xi} = -\mathbf{g} - \frac{ff_\xi + (1/2r)f(1-f)}{1-f^2}\mathbf{g}_\xi + \frac{f_\xi + (1/2r)(1-f)}{1-f^2}\mathbf{g}_\eta,$$

$$\mathbf{g}_{\eta\eta} = -\mathbf{g} + \frac{f_\eta - (1/2r)(1-f)}{1-f^2}\mathbf{g}_\xi + \frac{-ff_\eta + (1/2r)f(1-f)}{1-f^2}\mathbf{g}_\eta. \tag{61}$$

Taking into account equation (1), rewritten in the vectorial form:

$$\mathbf{g}_{\xi\eta} = -(\mathbf{g}_\xi\mathbf{g}_\eta)\mathbf{g} - \frac{1}{2r}(\mathbf{g}_\xi - \mathbf{g}_\eta), \tag{62}$$

and using (60)-(62) we obtain the equation for the function u ($f(\xi,\eta) = \cos u$):

$$u_{\xi\eta} + \sin u = \frac{1}{4r}\frac{u_\eta - u_\xi}{\cos^2\frac{u}{2}} - \frac{1}{8r^2}\tanh\frac{u}{2}(13 - \tanh^2\frac{u}{2}). \tag{63}$$

In terms of the variables r , t it became:

$$u_{tt} - u_{rr} + \sin u = \frac{1}{4r}\frac{u_r}{\cos^2\frac{u}{2}} - \frac{1}{8r^2}\tanh\frac{u}{2}(13 - \tanh^2\frac{u}{2}). \tag{64}$$

Thus, we received the deformation of the standard equation sin-Gordon. On would suggest that the equation (63) (or (64)) is integrable.

The alternative approach can be conclude as next. We represent the derivatives to the currents as [10,11]:

$$U_\xi = a_1 C + a_2\,[U, C] \quad , \quad V_\eta = b_1 C + b_2\,[V, C]\,, \tag{65}$$

where $C = [U, V]$, and coefficients a_i, b_i are functions ξ and η. Multiplying the first equation of the system (65) on V, and second equation on U and calculating the traces, we have ($\tau \equiv -(1/2)Tr(UV) = \cos u$):

$$a_2 = -\frac{\tau_\xi}{4(1-\tau^2)} - \frac{1}{8r}\frac{1}{1+\tau},$$

$$b_2 = \frac{\tau_\eta}{4(1-\tau^2)} + \frac{1}{8\tau}\frac{1}{1+\tau}. \tag{66}$$

Putting $U = iU_i\sigma_i$, $V = iV_i\sigma_i$, turn out from (61) to the vectorial system and multiplying the first equation in terms of scalar on $\mathbf{V}_\xi$, and second on $\mathbf{U}_\eta$, we find coefficients a_1 and b_1:

$$a_1 = \frac{\mathbf{U}_\xi\mathbf{V}_\xi}{2\sin^2 u} + \frac{a_2}{r}, \quad b_1 = -\frac{\mathbf{U}_\eta\mathbf{V}_\eta}{2\sin^2 u} - \frac{b_2}{r}. \tag{67}$$

7. In conclusion we note, that approach, proposed in this paper, can be easy transferred on the case vacuum equations of Ernst in the theory of gravitation [12]. Then the role of matrices g will play the symmetrical and real matrices of nondiagonal part of the metrics, the involutions and kinematical connections (14)-(20) will be changed also.

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